軽微なミス(と思われるもの)は減点し，90点満点.

フーリエ解析　中間試験（2016/12/16）

1　次の関数 $f(t)$ のラプラス変換 $F(s)$ を求めなさい（簡単な形にまとめることが望ましい）.　5×8=40

（1）$f(t)=e^{at}\cos\lambda t$　　（2）$f(t)=t\sinh\lambda t$

（3）$f(t)=t^2e^{-at}$　　（4）$f(t)=\displaystyle\int_0^t \sin\tau\cos(t-\tau)d\tau$

（5）$f(t)=\begin{cases}\sin t & (0<t<\pi)\\ 0 & (\pi<t)\end{cases}$　　（6）$f(t)=\begin{cases}0 & (0<t<a)\\ t & (a<t)\end{cases}$

（7）$f(t)=\cos(\omega t+\theta)$　　（8）$f(t)=\dfrac{1-\cos t}{t}$

2　次の関数 $F(s)$ のラプラス逆変換 $f(t)$ を求めなさい.　5×8=40

（1）$F(s)=\dfrac{e^{-\pi s}}{s^4}$　　（2）$F(s)=\dfrac{s-b}{s^2-a^2}$

（3）$F(s)=\dfrac{s+1}{(s-1)(s-2)}$　　（4）$F(s)=\dfrac{2s}{s^2+2s+5}$

（5）$F(s)=\dfrac{1}{s^2(s^2+2)}$　　（6）$F(s)=\dfrac{1}{s(s+2)^2}$

（7）$F(s)=\dfrac{1}{(s+1)(s^2+4)}$　　（8）$F(s)=\dfrac{s}{(s^2+a^2)^2}$

3　次の微分方程式を，与えられた条件下で，ラプラス変換を用いて解きなさい.　10

$y''-y'-2y=6 \quad y(0)=1 \quad y'(0)=0$

$Y(s)=$ ...　5

$y(t)=$　5

表 I	$f(t)=\mathcal{L}^{-1}[F]$	$F(s)=\mathcal{L}[f]$
(1)	1	$\frac{1}{s}$
(2)	t	$\frac{1}{s^2}$
(3)	$\frac{t^{n-1}}{(n-1)!}$ (n は自然数)	$\frac{1}{s^n}$
(4)	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
(5)	$e^{\lambda t}$	$\frac{1}{s-\lambda}$
(6)	$\cos\lambda t$	$\frac{s}{s^2+\lambda^2}$
(7)	$\sin\lambda t$	$\frac{\lambda}{s^2+\lambda^2}$

表 II	$f(t)=\mathcal{L}^{-1}[F]$	$F(s)=\mathcal{L}[f]$
(1)	$\lambda f(t)+\mu g(t)$	$\lambda F(s)+\mu G(s)$
(2)	$\frac{1}{\lambda}f\left(\frac{t}{\lambda}\right)$	$\lambda F(\lambda s)$
(3)	$U(t-\lambda)f(t-\lambda)=\begin{cases}0 & (0\leqq t<\lambda)\\ f(t-\lambda) & (t\geqq\lambda)\end{cases}$	$e^{-\lambda s}F(s)$
(4)	$e^{-\lambda t}f(t)$	$F(s+\lambda)$
(5)	$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}F(s)$
(6)	$-tf(t)$	$F'(s)$
(7)	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$

1.

(1) $\mathcal{L}\{e^{at}\cos\lambda t\} = \dfrac{s-a}{(s-a)^2+\lambda^2}$

(2) $\mathcal{L}\{t\sin\lambda t\} = -\dfrac{d}{ds}\mathcal{L}\{\sin\lambda t\} = -\dfrac{d}{ds}\cdot\dfrac{\lambda}{s^2+\lambda^2}$

$= \dfrac{2s\lambda}{(s^2+\lambda^2)^2}$

(別)

$\mathcal{L}\left\{t\times\dfrac{e^{i\lambda t}-e^{-i\lambda t}}{2i}\right\} = \dfrac{1}{2i}\left\{\dfrac{1}{(s-i\lambda)^2} - \dfrac{1}{(s+i\lambda)^2}\right\}$

$= \dfrac{1}{2i}\cdot\dfrac{s^2+i2\lambda s-\lambda^2-(s^2-i2\lambda s-\lambda^2)}{(s-i\lambda)^2(s+i\lambda)^2} = \dfrac{2s\lambda}{(s^2+\lambda^2)^2}$

(3) $\mathcal{L}\{t^2e^{-at}\} = \dfrac{2}{(s+a)^3}$

別 $= \dfrac{d^2}{ds^2}\dfrac{1}{s+a}$

$= \dfrac{d}{ds}\times\left(-\dfrac{1}{(s+a)^2}\right) = \dfrac{2}{(s+a)^3}$

(4) $\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\} = \mathcal{L}\{\sin\tau\}\mathcal{L}\{\cos\tau\}$

$= \dfrac{1}{s^2+1}\times\dfrac{s}{s^2+1^2} = \dfrac{s}{(s^2+1)^2}$ (2(8)の答えにもなる)

(5) $f(t) = \sin t\cdot(u(t)-u(t-\pi))$

$= \sin t\cdot u(t) - \sin t\cdot u(t-\pi)$

$= \sin t\cdot u(t) + \sin(t-\pi)\,u(t-\pi)$

$\therefore \mathcal{L}\{f(t)\} = \dfrac{1}{s^2+1^2}\cdot e^{-0s} + \dfrac{1}{s^2+1^2}\cdot e^{-\pi s} = \dfrac{1+e^{-\pi s}}{s^2+1}$

(6)

$$f(t) = t\,u(t-a) = \cancel{a}(t-a+a)\,u(t-a)$$
$$= (t-a)\,u(t-a) + a\,u(t-a)$$

$$\mathcal{L}\{f(t)\} = \frac{e^{-as}}{s^2} + \frac{ae^{-as}}{s}$$

(7) $f(t) = \cos\omega t \cdot \cos\theta - \sin\omega t \cdot \sin\theta$.

$$\mathcal{L}\{f(t)\} = \cos\theta \times \frac{s}{s^2+\omega^2} - \sin\theta \cdot \frac{\omega}{s^2+\omega^2}$$

(8) $$\mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \int_s^{\infty} \mathcal{L}\{1-\cos t\}\,d\tilde{s} = \int_s^{\infty}\left\{\frac{1}{\tilde{s}} - \frac{\tilde{s}}{\tilde{s}^2+1}\right\}d\tilde{s}$$

$$= \left[\log\tilde{s} - \frac{1}{2}\log(\tilde{s}^2+1)\right]_s^{\infty} = \left[\log\frac{\tilde{s}}{\sqrt{\tilde{s}^2+1}}\right]_s^{\infty} = \log\frac{\sqrt{s^2+1}}{s}$$

$$\left(= \frac{1}{2}\log\left(1+\frac{1}{s^2}\right)\right)$$

2. (1) $$F(s) = \frac{e^{-\pi s}}{s^4} \xrightarrow{\mathcal{L}^{-1}} f(t) = \frac{1}{6} \times (t-\pi)^3 u(t-\pi)$$

$$\frac{1}{s^4} \xrightarrow{\mathcal{L}^{-1}} \frac{t^3}{3!}$$

$$= \begin{cases} 0 & 0<t<\pi \\ \frac{1}{6}(t-\pi)^3 & \pi<t \end{cases}$$

(2) $$F(s) = \frac{s-b}{s^2-a^2} = \frac{s}{s^2-a^2} - \frac{b}{a}\frac{a}{s^2-a^2} \xrightarrow{\mathcal{L}^{-1}} \underline{f(t) = \cosh at - \frac{b}{a}\sinh at}$$

or

$$\frac{s-b}{(s+a)(s-a)} = \frac{A}{s+a} + \frac{B}{s-a} \xrightarrow{\mathcal{L}^{-1}} \underline{f(t) = Ae^{-at} + Be^{at}}$$

$$A = \frac{-a-b}{-2a} = \frac{a+b}{2a} \quad B = \frac{a-b}{2a}$$

(3) $F(s) = \dfrac{s+1}{(s-1)(s-2)} = \dfrac{A}{s-1} + \dfrac{B}{s-2}$

$A(s-2)+B(s-1)$

$A+B=1 \quad -2A-B=1$

$$\begin{array}{r} -2A+B=1 \\ \hline -A \quad =2 \end{array}$$

$A = F(s)(s-1)\Big|_{s=1} = \dfrac{1+1}{1-2} = -2$

$B = F(s)(s-2)\Big|_{s=2} = \dfrac{2+1}{2-1} = 3$

$\therefore f(t) = -2e^{t} + 3e^{2t}$

(4) $F(s) = \dfrac{2s}{s^2+2s+5} = \dfrac{2(s+1-1)}{(s+1)^2+2^2} = \dfrac{2(s+1)}{(s+1)^2+2^2} - \dfrac{2}{(s+1)^2+2^2}$

$\xrightarrow{\mathcal{L}^{-1}} f(t) =$

$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$

(5) $F(s) = \dfrac{1}{s^2(s^2+2)} = \dfrac{1}{2}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+2}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{s^2} - \dfrac{1}{\sqrt{2}}\dfrac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right) \xrightarrow{\mathcal{L}^{-1}} f(t) = \dfrac{1}{2}\left(t - \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)$

$\dfrac{1}{s^2+2} = \dfrac{1}{\sqrt{2}}\dfrac{\sqrt{2}}{s^2+(\sqrt{2})^2} \xrightarrow{\mathcal{L}^{-1}} \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t$

$\dfrac{1}{\sqrt{2}}\displaystyle\int_0^t \sin\sqrt{2}\tilde{t}\,d\tilde{t} = \dfrac{1}{\sqrt{2}}\left[\dfrac{-1}{\sqrt{2}}\cos\sqrt{2}\tilde{t}\right]_0^t = \dfrac{1}{2}(1-\cos\sqrt{2}t)$

$\dfrac{1}{2}\displaystyle\int_0^t (1-\cos\sqrt{2}\tilde{t})\,d\tilde{t} = \dfrac{1}{2}\left[\tilde{t} - \dfrac{1}{\sqrt{2}}\sin\sqrt{2}\tilde{t}\right]_0^t$

$= \dfrac{1}{2}\left(t - \dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)$

(6)

$$F(s) = \frac{1}{s(s+2)^2} = \frac{1}{s}\mathcal{L}\{e^{-2t}t\}$$

$$\therefore f(t) = \int_0^t e^{-2\tau}\tau\, d\tau = \left[-\frac{1}{2}e^{-2\tau}\cdot\tau\right]_0^t + \frac{1}{2}\int_0^t e^{-2\tau} d\tau$$

$$= -\frac{e^{-2t}}{2}t + \frac{1}{2}\left[-\frac{e^{-2\tau}}{2}\right]_0^t$$

$$= -\frac{te^{-2t}}{2} + \frac{1}{4}(1-e^{-2t})$$

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \xrightarrow{\mathcal{L}^{-1}} f(t) = A + Be^{-2t} + C\cdot e^{-2t}\cdot t$$

$$A = F(s)s\Big|_{s=0} = \frac{1}{4} \qquad B = \frac{d}{ds}F(s)(s+2)^2\Big|_{s=-2}$$

$$= \frac{d}{ds}\left(\frac{1}{s}\right)\Big|_{s=-2} = -\frac{1}{s^2}\Big|_{s=-2} = -\frac{1}{4}$$

$$C = F(s)(s+2)^2\Big|_{s=-2} = -\frac{1}{2}$$

(7)

$$F(s) = \frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2^2}$$

$$= \frac{A(s^2+2^2) + (Bs+C)(s+1)}{(s+1)(s^2+2^2)}$$

s^2: $A + B = 0$ ①

s^1: $B + C = 0$ ②

s^0: $4A + C = 1$ ③

① − ②

$$\begin{array}{rl} A - C &= 0 \\ 4A + C &= 1 \\ \hline 5A &= 1 \end{array}$$

$A = \frac{1}{5}$ $B = -\frac{1}{5}$

$$C = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\therefore F(s) = \frac{1}{5}\left(\frac{1}{s+1} - \frac{s-1}{s^2+2^2}\right) = \frac{1}{5}\left(\frac{1}{s+1} - \frac{s}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+2^2}\right)$$

$$\therefore f(t) = \frac{1}{5}\left(e^{-t} - \cos 2t + \frac{1}{2}\sin 2t\right)$$

$$A = F(s)(s+1)\Big|_{s=-1} = \frac{1}{5}$$

合成積として考えると

$$\frac{1}{s+1}\cdot\frac{1}{s^2+2^2} = \mathcal{L}\{e^{-t}\}\mathcal{L}\left\{\frac{1}{2}\sin 2t\right\}$$

$$= \frac{1}{2}\int_0^t \sin 2\tau \cdot e^{-(t-\tau)} d\tau = \frac{e^{-t}}{2}\underbrace{\int_0^t e^{\tau}\sin 2\tau d\tau}_{I}$$

$$I = \left[e^{\tau}\sin 2\tau\right]_0^t - 2\int_0^t e^{\tau}\cos 2\tau d\tau = e^t\sin 2t$$

$$-2\left\{\left[e^{\tau}\cos 2\tau\right]_0^t + 2\int_0^t e^{\tau}\sin 2\tau d\tau\right\}$$

$$= e^t\sin 2t - 2\left\{(e^t\cos 2t - 1) + 2I\right\}$$

$$= e^t\sin 2t - 2e^t\cos 2t + 2 - 4I.$$

$$\therefore I = \frac{1}{5}\left(e^t\sin 2t - 2e^t\cos 2t + 2\right)$$

$$\therefore f(t) = \frac{e^{-t}}{2}\times\frac{1}{5}\left(e^t\sin 2t - 2e^t\cos 2t + 2\right)$$

$$= \frac{1}{5}\left(\frac{1}{2}\sin 2t - \cos 2t + e^{-t}\right)$$

(8)

$$F(s) = -\frac{d}{ds}\left(\frac{1}{2}\frac{1}{s^2+a^2}\right) = -\frac{d}{ds}\left(\underbrace{\frac{1}{2a}\cdot\frac{a}{s^2+a^2}}_{\mathcal{L}\left\{\frac{1}{2a}\sin at\right\}}\right)$$

$$\therefore f(t) = \frac{1}{2a}t\cdot\sin at.$$

合成積とすると

$$\frac{1}{a}\frac{s}{s^2+a^2}\cdot\frac{a}{s^2+a^2} = \frac{1}{a}\int_0^t \underbrace{\overset{A}{\sin a\tau}\cdot\overset{B}{\cos a(t-\tau)}}_{\frac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)}d\tau.$$

$$= \frac{1}{2a}\int_0^t\left(\sin a(\tau+t-\tau)+\sin a(\tau-t+\tau)\right)d\tau$$

$$= \frac{1}{2a}\int_0^t\left(\sin at+\sin a(2\tau-t)\right)d\tau$$

$$= \frac{1}{2a}\left[(\sin at)\tau - \frac{1}{2a}\cos a(2\tau-t)\right]_0^t$$

$$= \frac{1}{2a}\left[\left[t\cdot\sin at - \frac{1}{2a}\left[\cos a\cdot t - \cos a(-t)\right]\right]\right]$$

$$= \frac{1}{2a}\cdot t\sin at$$

3 $\quad y'' - y' - 2y = 6 \qquad y(0) = 1,\ y'(0) = 0$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - s$$

$$\mathcal{L}\{y'\} = s\,Y(s) - y(0) = s\,Y(s) - 1$$

$$(s^2 Y(s) - s) - (s\,Y(s) - 1) - 2Y(s) = \frac{6}{s}$$

$$(s^2 - s - 2)Y(s) - s + 1 = \frac{6}{s}$$

$$(s+1)(s-2)Y(s) = \frac{6}{s} + s - 1$$

$$\therefore Y(s) = \frac{6}{s(s+1)(s-2)} + \frac{s-1}{(s+1)(s-2)}$$

$$= Y_1(s) + Y_2(s)$$

$$Y_1(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2} \qquad A = Y_1(s)\cdot s\Big|_{s=0} = \frac{6}{1\cdot(-2)} = -3$$

$$\downarrow \qquad B = Y_1(s)(s+1)\Big|_{s=-1} = \frac{6}{-1\cdot -3} = 2$$

$$y_1(t) = A + Be^{-t} + Ce^{2t} \qquad C = Y_1(s)(s-2)\Big|_{s=2} = \frac{6}{2\cdot 3} = 1.$$

$$Y_2(s) = \frac{D}{s+1} + \frac{E}{s-2} \qquad D = Y_2(s)(s+1)\Big|_{s=-1} = \frac{-2}{-3} = \frac{2}{3}$$

$$y_2(t) = De^{-t} + Ee^{2t} \qquad E = Y_2(s)(s-2)\Big|_{s=2} = \frac{1}{3}$$

$$\therefore y(t) = -3 + \left(2 + \frac{2}{3}\right)e^{-t} + \left(1 + \frac{1}{3}\right)e^{2t}$$

$$= -3 + \frac{8}{3}e^{-t} + \frac{4}{3}e^{2t}$$

$$y(0) = -3 + \frac{12}{3} = -3 + 4 = 1.$$

$$y'(t) = -\frac{8}{3} + \frac{8}{3} e^{2t} \qquad y'(0) = 0$$

e^{-t}