A METHOD FOR ESTIMATING PROFILE LOSS OF LOW PRESSURE TURBINE BLADES FROM THE LOW SPEED CASCADE TEST DATA

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ABSTRACT
This paper describes the method for accurately estimating profile loss using measured flow parameters in a low speed cascade test. One of the issues of the control volume analysis for obtaining the mixed-out profile loss is to give realistic flow conditions on the control surface at the trailing edge plane. In this paper, the flow conditions at the trailing edge plane are related to the measured velocity distribution on the suction surface from throat to the trailing edge using the concept of circulation.

The validity of the estimation method is verified by using the results of steady Reynolds Averaged Navier-Stokes (RANS) simulation. In this verification, the verification profile loss of the RANS simulation is obtained by using the calculated drag forces of surface pressure and surface shear stress. The verification is conducted for two kinds of blade profiles at three different Reynolds numbers, 57,000, 100,000 and 147,000, for each blade profile. It is found that all the estimations by the current method are in a range between -10% to +6% of the verification profile losses of the RANS simulations.

This method is applied to the data analysis of low speed cascade tests for the two kinds of blade profiles used in the verification. It is shown that the measured pressure loss downstream of the cascade includes an additional loss to the profile loss even in the steady flow.

INTRODUCTION
Both the efficiency and the weight of low-pressure turbines (LPT) have a large effect on the fuel burn of a high bypass ratio turbofan engine. For this reason, the efforts for LPT improvement have been made not only to reduce LPT pressure losses but also reduce the number of the blades (increasing the blade lift). Because of a high aspect ratio design for LP turbines of a high bypass ratio turbofan engine, profile loss, which is a two-dimensional loss generated in the blade boundary layers, is the largest loss in the LPT pressure losses. Consequently, most of the works have concentrated on the reduction in the profile loss. However, the profile loss generally increases with increasing the blade lift. In case of high lift cascades with a high velocity deceleration rate in the rear part of the blade (aft loaded case), a higher blade lift leads to a risk of significantly high profile loss [1]. This makes it difficult to achieve the increase in the efficiency and the reduction in the weight of LP turbines simultaneously. Because of a continuous demand for improving propulsive efficiency, further growth of bypass ratio will be required. Therefore, the efforts to reduce the profile loss for higher lift blades are still needed.

Low speed cascade tests have been widely used to experimentally investigate the loss generation mechanism of profile losses, because a large scale of blades enable us to measure the flow structures of blade boundary layers in detail. A two-dimensional feature of profile loss also enables us to conduct the experiments using a simple rectilinear cascade. In order to investigate the unsteady effects of incoming wakes on the development of blade boundary layers, a moving-bar mechanism has been used to simulate the incoming wakes. Schulte and Hodson [2] examined the influences of wake-passing frequency and wake strength on the profile loss. Steiger and Hodson [3] observed the wake-induced vortices formed in the separated shear layer on the suction surface convecting through the boundary layer. It is considered that the formation and succeeding breakdown of the wake-induced vortices might...
lead to another source of profile loss due to unsteady wake interaction [4].

In the experiments, the total pressure loss is obtained from the inlet total pressure measurement and the total pressure measurement downstream of the cascade. However, the total pressure loss downstream of the cascade doesn’t always represent the profile loss of the cascade. Especially, in an unsteady flow, the mixing loss of incoming wakes generated in the cascade passage is included in the total pressure loss downstream of the cascade. It is necessary to estimate the profile loss by using the other measurement data. The detailed boundary layer measurement data obtained by low speed cascade tests is very useful for the profile loss estimation.

Using a control volume analysis, Denton [5] derived the expression for the mixed-out profile loss in which a momentum deficit loss is expressed by the boundary layer integral parameters. This method is very useful, because the profile loss is expressed in an explicit form. Therefore, this method has been widely used for the profile loss estimation in the low speed cascade tests (e.g. [6], [7]). One of the issues of the control volume analysis for obtaining the downstream mixed-out flow is to give realistic free stream flow parameters on the control surface at the trailing edge plane. In the low speed cascade tests, it is possible to obtain the flow parameters by detailed measurement across the free stream flow along the trailing edge plane, as well as across the blade boundary layers. But it is very time consuming. For the performance evaluation by a low speed cascade test, it is required to perform the measurements over a range of Reynolds numbers and wake interaction frequencies with different free stream turbulence levels. It is not practical to conduct time consuming measurement for each test condition. It would be extremely helpful to estimate the flow conditions at the trailing edge plane from the accompanying measurement data, such as blade surface static pressures.

This paper aims at developing the method for accurately estimating the profile loss using measured flow parameters in a low speed cascade test. In this paper, using the concept of circulation, the flow conditions at the trailing edge plane are related to the measured velocity distribution on the suction surface from throat to the trailing edge. The validity of the estimation method is verified by using the results of steady Reynolds Averaged Navier-Stokes (RANS) simulation. This method is applied to the estimation of the profile loss for low speed cascade tests. The estimated profile loss implies that the measured pressure loss downstream of the cascade includes an additional loss to the profile loss even in the steady flow.

**NOMENCLATURE**

- \( C_p \) Static pressure coefficient defined by Eq. (27)
- \( F_{d,x} \) Skin friction drag force in the axial direction
- \( F_{d,y} \) Skin friction drag force in the tangential direction
- \( F_{p,x} \) Pressure force in the axial direction
- \( F_{p,y} \) Pressure force in the tangential direction
- \( f_{d,y} \) Local shear stress in the tangential direction
- \( \dot{m} \) Mass flow rate
- \( P_{T1} \) Upstream total pressure
- \( p_i \) Upstream static pressure
- \( p_b \) Local base pressure
- \( \bar{p}_b \) Average base pressure over the trailing edge circle
- \( p_e \) Downstream mixed-out static pressure
- \( p_s \) Blade surface static pressure
- \( p_{te} \) Local static pressure on the trailing edge plane \( AB \) in Fig. 1
- \( \bar{p}_{te} \) Average static pressure over the trailing edge plane \( AB \) in Fig. 1
- \( \Delta P_T \) Total pressure loss
- \( R_e \) Reynolds number based on chord length and exit average velocity
- \( S \) Pitch
- \( S_{bx} \) Axial length between \( B \) and \( C \) in Fig. 1
- \( S_{gy} \) Tangential length between \( B \) and \( C \) in Fig. 1
- \( S_e \) Length of line \( AB \) in Fig. 1
- \( S_t \) Strouhal number based on chord length and inlet velocity
- \( s \) Blade surface distance
- \( t \) Trailing edge thickness or coordinate in Fig. A-1
- \( V_1 \) Upstream velocity
- \( V_e \) Downstream mixed-out velocity
- \( V_{te} \) Average velocity over the trailing edge plane \( AB \) in Fig. A-1
- \( V_c \) Axial velocity
- \( \dot{V}_b \) Velocity associated with base pressure defined in Eq. 28
- \( \bar{V}_m \) Mixed-out velocity at the trailing edge plane (see ANNEX B)
- \( v_{te} \) Local velocity on the trailing edge plane \( AB \) in Fig. 1
- \( W_r \) Width defined by Eq. (A-3) in ANNEX A
- \( w \) Coordinate in Fig. A-1 in ANNEX A
- \( x \) Axial coordinate in Fig. 1
- \( y \) Tangential coordinate in Fig. 1
- \( Z \) Zweifel number defined by Eq. 25
- \( \alpha_e \) Angle formed by the trailing edge plane and the tangent plane in Fig. 1
- \( \beta_1 \) Upstream flow angle
- \( \beta_e \) Downstream mixed-out flow angle
- \( \beta_{te} \) Local flow angle on the trailing edge plane \( AB \) in Fig. 1
- \( \bar{\beta}_{te} \) Average flow angle on the trailing edge plane \( AB \) in Fig. 1
- \( \bar{\beta}_m \) Mix-out flow angle at the trailing edge plane (see ANNEX C)
- \( \delta \) Displacement thickness defined by Eq. (A-5)
- \( \theta \) Momentum thickness defined by Eq. (A-9)
- \( \rho_0 \) Density
- \( \omega \) Total Pressure loss coefficient defined by Eq. 37
FORMULATION OF CONTROL VOLUME ANALYSIS

Conservation Equations of Mass and Momentum

Consider the control volume ABCDE, as shown in Fig. 1, for estimating mixed-out velocity $V_e$, flow angle $\beta_e$ and static pressure $p_e$ downstream of the cascade. The flow enters across the control surface at the trailing edge plane AB. Here, A is the suction surface trailing edge point and B is the pressure surface trailing edge point of the adjacent blade. $p_b$ is the given average pressure acting on the base of the trailing edge BC, and $\alpha_{te}$ is the blade metal angle at the trailing edge. The angles of the control surface CD and EA to the axial plane are the same as the average flow angle $\beta_{te}$ of the flow at the trailing edge plane AB. Therefore, it can be assumed that the flows on the control surfaces CD and EA are periodic.

We apply the conservation equations of mass and momentum to the control volume ABCDE per unit span.

Mass conservation

$$\dot{m} = \rho_0 V_e S \cos(\beta_e)$$  \hspace{1cm} (1)

Momentum conservation in the axial direction

$$\dot{m} V_e \cos(\beta_e) - \int_0^{S_e} \rho_0 V_{te} \cos(\beta_{te} + \alpha_e) V_{te} \cos(\beta_{te}) \, dt$$

Momentum conservation in the tangential direction

$$\dot{m} V_e \sin(\beta_e) - \dot{m} V_{te} \sin(\beta_{te}) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}}$$

$$= \bar{p}_{te}(S - S_{by}) + \bar{p}_b S_{by} - p_e S$$  \hspace{1cm} (5)

Momentum conservation in the tangential direction

$$\dot{m} V_e \sin(\beta_e) - \dot{m} V_{te} \sin(\beta_{te}) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}}$$

$$= (\bar{p}_b - \bar{p}_{te}) S_{bx}$$  \hspace{1cm} (6)

Here, $V_{te}$ is the average velocity, $\delta$ is the displacement thickness, and $\theta$ is the momentum thickness. The mixed-out flow parameters are obtained by solving Eq. (4), (5) and (6) with known flow parameters at the inflow control surface.

As shown in ANNEX B, the expression equivalent to Denton’s profile loss equation [5] can be derived from Eq. (4), (5) and (6) by idealizing the flow model further as follows.

(a) The flow on the trailing edge plane AB is kept the same until the trailing edge tangential plane.

(b) The difference between $\beta_{te}$ and $\beta_e$ is small.

Coull [8] compared the mixed-out profile losses calculated using Mises simulations with the predicted profile losses by the Denton’s equation for the Mises results, and observed that there were differences in the overall levels of up to 18%. He suggested...
that the variance is probably due to the assumptions underpinning the Denton’s equation, such as zero deviation angle. The present method is considered to estimate the profile loss with less impact of the underpinning assumptions than the Denton’s method.

**Fig. 2 Control Volume for Circulation**

**Estimation of the Flow Parameters at TE plane**

In case of a low speed cascade test without detailed measurement for the flow at the trailing edge plane, it is necessary to estimate the trailing edge flow from the accompanying measurement data, such as blade surface static pressures. From the measured surface static pressures, the surface distribution of the inviscid velocity at the boundary layer edge is easily calculated. For an inviscid flow, the surface velocity distribution can be related to the lift by using the concept of circulation. Here, we attempt obtaining the flow conditions at the trailing edge plane from the surface velocity distributions using the concept of circulation.

Consider the closed contour $ABCD$ as shown in Fig. 2. The paths $AB$ and $CD$ are a tangential line with the tangential length equal to the pitch located at far upstream and far downstream, respectively. The paths $BC$ and $DA$ are located at mid-pitch and have the same contour. It can be assumed that the flows on the paths $BC$ and $DA$ are periodic, so the line integral of the velocity tangent to the paths $BC$ and $DA$ offset each other. The resultant circulation around the contour $ABCD$ can be expressed as follows:

$$\Gamma_{ABCD} = V_1 \sin(\beta_1) S + V_e \sin(\beta_e) S \quad (7)$$

**Fig. 3 Control Volume for Circulation inside Passage**

The circulation around the contour of the airfoil is:

$$\Gamma_{airfoil} = -\int_{LE}^{TE} V_{s, pressure} ds + \int_{LE}^{TE} V_{s, suction} ds \quad (8)$$

$$V_s = \sqrt{\frac{2(P_{T1} - p_s)}{\rho_0}} \quad (9)$$

Here, $P_{T1}$ is the upstream total pressure and $p_s$ is the blade surface static pressure. $V_{s, pressure}$ and $V_{s, suction}$ are the free stream velocities at the boundary layer edge, expressed by Eq. (9), on the pressure surface and the suction surface, respectively. The concept of circulation is an inviscid theory. However, it can be a good approximation to real viscous flow for typical aerodynamic applications. In the inviscid theory, the circulation is the same around every circuit. Therefore, it can be assumed that $\Gamma_{ABCD}$ is approximately equal to $\Gamma_{airfoil}$.

$$-\int_{LE}^{TE} V_{s, pressure} ds + \int_{LE}^{TE} V_{s, suction} ds = V_1 \sin(\beta_1) S + V_e \sin(\beta_e) S \quad (10)$$

The momentum conservation in the tangential direction is expressed as:

$$\dot{m}V_e \sin(\beta_e) - \dot{m}V_1 \sin(-\beta_1) = F_{p,y} + F_{d,y} \quad (11)$$

Here, $F_{p,y}$ is the total tangential component of the pressure force acting on the fluid per unit span.
\[ F_{p,y} = \int_0^{C_x} p_{s,upper} ds + \int_0^{C_x} p_{s,lower} ds \]  

(12)

\[ F_{d,y} \] is the total tangential component of the skin friction drag force acting on the fluid per unit span.

\[ F_{d,y} = \int_0^{S_{upper}} f_{d,y} ds + \int_0^{S_{lower}} f_{d,y} ds \]  

(13)

Here, \( f_{d,y} \) is the tangential component of local shear stress per unit span, and \( S_{upper} \) and \( S_{lower} \) are the surface length of the upper and the lower surface, respectively. From Eq. (10) and (11), we can assume the relation between the circulation and the blade lift due to the drag forces of surface pressure and surface skin friction, as follows:

\[
- \int_{LE}^{TE} V_{s,pressure} ds + \int_{LE}^{TE} V_{s,suction} ds = \frac{F_{p,y} + F_{d,y}}{\rho_0 V_x}
\]  

(14)

Next, we evaluate the circulation around the closed contour \( ABCDEF \) shown in Fig. 3. The path \( AB \) is a tangential line located at far upstream with the tangential length equal to the pitch. The angles of the paths \( BC \) and \( FA \) to the axial plane are the same. Because the flows on the paths \( BC \) and \( FA \) can be assumed to be periodic, the line integral of velocity tangent to the paths \( BC \) and \( FA \) offset each other. The path \( DE \) is the throat line to which the flow is perpendicular. Therefore, the line integral of velocity tangent to the path \( DE \) is zero. Because there is no body force inside the contour \( ABCDEF \), the circulation around the contour is zero. Using Eq. (8), this leads to the following relation:

\[ V_t \sin \beta_1 S = \Gamma_{airfoil} - \int_{LE}^{TE} V_{s,suction} ds \]  

(15)

Similarly, we consider another closed contour \( ABCDGEF \) in Fig. 3. The path \( DG \) lies on the trailing edge plane from the pressure surface trailing edge point \( D \) to the suction surface trailing edge point \( G \) of the adjacent blade. For the same reason as the circulation around the contour \( ABCDEF \), the circulation around the contour \( ABCDGEF \) is also zero. From this relation, the line integral of velocity tangent to the trailing edge plane \( DG \) can be expressed as follows:

\[
\int_0^{S_e} v_t \sin(\beta_{te} + \alpha_e) dt = -V_t \sin(\beta_1) S + \Gamma_{airfoil}
\]

(16)

Now, the flow condition at the trailing edge plane is related to the measured velocity distribution on the suction surface from throat to the trailing edge. It is necessary to derive the average velocity \( V_{te} \) and the average flow angle \( \beta_{te} \) using the relation expressed by Eq. (16).

Following the flow modelling at the trailing edge plane described in ANNEX A, Equation (16) can be rewritten as follows:

\[ V_{te} \sin(\beta_{te} + \alpha_e) S_e \left( 1 - \frac{\delta}{W_e} \right) = \int_{Throat}^{TE} V_{s,suction} ds \]  

(17)

The displacement thickness in Eq. (17) is obtained by using the integral of the velocity over the trailing edge plane, not only in the boundary layer region but also in the free stream region as seen in Eq. (A.5) in ANNEX A. It should be noted that the displacement thickness by this definition is not necessarily equal to the displacement thickness which is the sum of the boundary layer displacement thickness at the trailing edge on the pressure surface \( \delta_m,pressure \) and that at the trailing edge on the suction surface \( \delta_m, suction \), because the real free stream flow is typically nonuniform. Similarly, the momentum thickness defined by Eq. (A.9) in ANNEX A is not necessarily equal to the sum of the boundary layer momentum thickness at the trailing edge on the pressure surface \( \theta_m,pressure \) and that at the trailing edge on the suction surface \( \theta_m, suction \). However, in the present analysis, only the measured boundary layer integral parameters are available. It is necessary to obtain the average flow parameters by approximating the flow expressed by using the measured boundary layer integral parameters. The best approximation is that the approximated flow has the same mass flow and the same momentum flux as those of the real flow at the trailing edge plane.

The mass flow across the trailing edge plane \( DG \) in Fig. 3 is expressed as follows:

\[ m = \rho_0 V_t \cos(\beta_{te} + \alpha_e) S_e \left( 1 - \frac{\delta}{W_e} \right) \]  

(18)

On the other hand, as shown in ANNEX C, the momentum flux tangent to the trailing edge plane is expressed as follows:

\[ m V_{te} \sin(\beta_{te} + \alpha_e) \frac{1 - \delta}{W_e} \frac{\theta}{W_e} = m \int_{Throat}^{TE} V_{s,suction} ds \]  

(19)

Combining Eq. (18) with (19), the following relation is given:

\[ \tan(\beta_{te} + \alpha_e) = \frac{\rho_0 \left( 1 - \frac{\delta}{W_e} \right)^2}{m} \int_{Throat}^{TE} V_{s,suction} ds \]  

(20)
Using Eq. (20) with $\delta = \delta_{m,p} + \delta_{m,s}$ and $\theta = \theta_{m,p} + \theta_{m,s}$, the average flow angle $\beta_{te}$ can be obtained. And then, by using this $\beta_{te}$, the average velocity $V_{te}$ of the approximated flow can be obtained from Eq. (18).

### Procedure of Control Volume Analysis

In case of a control volume analysis for a low speed cascade test without detailed measurement for the free stream flow, the mass flow rate is an unknown parameter. In the present analysis, the mass flow rate is a solution to be obtained by solving the conservation equations of mass and momentum. The upstream flow angle $\beta_{1}$ appeared in Eq. (7) etc. is also an unknown parameter. In an unsteady flow, the inlet flow angle changes due to the movement of the bars. By using unsteady Direct Numerical Simulations coupled with moving bars, Michelassi et al. [9] predicted the differences in the inlet flow angle for unsteady flows and showed the effect on the surface static pressure distribution and the wall shear stress. In the present analysis, the upstream flow angle is estimated by using the measured static pressure distribution.

First of all, a provisional mass flow rate is given. By using the measured blade surface velocities, the tangential component of the upstream velocity can be obtained from Eq. (15). Knowing the mass flow rate and the tangential velocity, the upstream velocity $V_{1}$ and the upstream flow angle $\beta_{1}$ are determined. From Eq. (10) and (14), the downstream mixed-out tangential velocity can be expressed as follows:

$$V_{e} \sin(\beta_{e}) = \frac{F_{p,y} + F_{d,y}}{m} - V_{1} \sin(\beta_{1})$$

(21)

From Eq. (12), the total tangential component of the pressure force acting on the fluid per unit span $F_{p,y}$ can be obtained by using the measured static pressure on the blade surfaces. Here, we assume that the total tangential component of the skin friction drag force acting on the fluid per unit span $F_{d,y}$ is known. Using these tangential forces, $V_{e} \sin(\beta_{e})$ is obtained from Eq. (21). And then the downstream mixed-out velocity $V_{e}$ and flow angle $\beta_{e}$ can be obtained by using the relation of mass conservation that $\rho_{0}V_{e}\cos(\beta_{e})$ is equal to $\rho_{0}V_{1}\cos(\beta_{1})$. From Eq. (19), the second term on the left-hand side of Eq. (6) can be expressed using the mass flow rate and the line integral of velocity tangent to the suction surface from throat to the trailing edge. Then, from Eq. (6), the average static pressure at the trailing edge plane $\bar{p}_{te}$ can be obtained by using measured $\bar{p}_{b}$.

$$\bar{p}_{te} = \bar{p}_{b} - \frac{m}{S_{bx}} \left( V_{e} \sin(\beta_{e}) - \frac{\sin(\beta_{te})}{\sin(\beta_{te} + \alpha_{e})} \int_{Throat}^{TE} \frac{V_{s,suction}}{S_{s}} ds \right)$$

(22)

On the other hand, because the total pressure at the trailing edge plane can be assumed to be the same as the known upstream total pressure $P_{T1}$, the average static pressure at the trailing edge plane $\bar{p}_{te}$ can be obtained using the average velocity $V_{te}$ obtained from Eq. (18) and (20) with the measured boundary layer integral parameters as follows:

$$\bar{p}_{te} = P_{T1} - \frac{1}{2} \rho_{0} V_{te}^{2}$$

(23)

The present problem is to find the mass flow rate by which $\bar{p}_{te}$ from Eq. (22) corresponds to $\bar{p}_{te}$ from Eq. (23).

Finally, the downstream static pressure $p_{e}$ can be obtained from Eq. (5) with the measured boundary layer integral parameters. Consequently, the mixed-out profile loss is expressed as follows:

$$\Delta P_{T} = P_{T1} - \left( p_{e} + \frac{1}{2} \rho_{0} V_{e}^{2} \right)$$

(24)

### VERIFICATION OF VALIDITY OF THE METHOD

In this method for estimating the profile loss, some assumptions are made to simplify the flow model. And the assumptions based on the inviscid theory of circulation are also made to approximate a real viscous flow. It is necessary to verify the validity of the assumptions. Here, we use the results of steady Reynolds Averaged Navier-Stokes (RANS) simulations for the verification. And for the evaluation of the accuracy of profile loss estimation, we also use the verification profile loss of the RANS simulations which are obtained by using the calculated drag forces of surface pressure and surface skin friction.

<table>
<thead>
<tr>
<th>Blade Design</th>
<th>Axial Chord Cx (mm)</th>
<th>Cx based Solidity (Cx / S)</th>
<th>Zweifel Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade A</td>
<td>100</td>
<td>1.149</td>
<td>1.109</td>
</tr>
<tr>
<td>High diffusion on the suction surface</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blade B</td>
<td>100</td>
<td>1.149</td>
<td>0.859</td>
</tr>
<tr>
<td>Full laminar flow on the suction surface</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1 Blade Design Parameters

#### Cascade Details

The verification is performed for two kinds of blade design with different types of suction surface velocity distributions. One has typical velocity distribution in which the flow remains laminar over a significant fraction of the suction surface and undergoes transition via a laminar separation bubble. It is designed to locate the point of peak velocity as far back as possible while ensuring reattachment of the laminar separation before the trailing edge. The axial chord solidity of 1.149 is about 15 percent lower than the axial chord solidity of the lowest pressure loss. This blade is referred to as Blade A.

Another blade has suction surface velocity distribution in which a lammar flow is maintained until the trailing edge. This blade is referred to as Blade B. The axial chord solidity and the tangential momentum change across the cascade are the same as those of Blade A, but the exit to inlet velocity ratio of Blade B is selected higher than that of Blade A to make the suction surface...
diffusion lower. The design parameters are compared in Table 1. Here, the Zweifel number, \( Z \), is defined as follows:

\[
Z = \frac{2S}{C_x} \left( \frac{\tan(\beta_e) + \tan(\beta_1)}{\sec \beta_e^2} \right)
\]  

(25)

A higher exit to inlet velocity ratio design of Blade B results in a lower Zweifel number compared to Blade A.

**Cascade Flow Conditions**

The purpose of the present study is to verify the validity of the method for estimating a profile loss. For this reason, we choose a steady flow for the verification to eliminate the other sources of pressure loss.

The verification is performed at three different Reynolds numbers, 57,000, 100,000 and 147,000, for the above two kinds of blade profiles. It is well-known that the impact of Reynolds number on the profile loss is significant, especially for the case with a strong deceleration rate on the suction surface (Blade A). The pressure loss due to a laminar separation bubble becomes significantly high at low Reynolds numbers [4]. The range of the selected Reynolds numbers covers this flow condition.

**Numerical Method**

The present study employs a commercial flow solver, ANSYS CFX Ver.15(ANSYS) to execute steady RANS simulation for the verification. SST (Shear-Stress Transport) turbulence model with a transition model is used, which is vital for achieving a good agreement between the experimental data and the numerical results [10]. A velocity vector and inlet turbulence intensity are specified on the inflow boundary and a constant static pressure is given on the outflow boundary. The spanwise extension of the domain is 15 percent axial chord. The grid is generated by using Pointwise Ver.16.04R4 (Pointwise). The grid system consists of H-O-H type sub-systems so as to attain a high-quality grid generation around the airfoil. Accordingly, the closest point to the surface is located so that \( y^+ \) of the point is less than unity. The total numbers of the grid points are about 5.7 million for Blade A simulation, and about 5.5 million for Blade B simulation.

**Calculated Base Pressure**

From the results of steady RANS simulations, the surface static pressures are obtained at the grid points on the blade surface. The number of the grid points are 733 for Blade A and 570 for Blade B. Figure 4 shows the grid points around the trailing edge circle. The base pressure is defined as the pressure acting on the tailing edge circle in an opposite direction of vector \( \vec{h} \) in Fig. 4. The average base pressure \( \bar{p}_b \) can be expressed as follows:

\[
\bar{p}_b = \frac{1}{t} \int_0^t p_x \, dg
\]

(26)
The profile features of the static pressure distribution around the trailing edge at Re = 57,000 can be seen in Fig. 5 and 6 for Blade A and B, respectively. The distribution profiles at different Reynolds numbers are similar for each Blade. Here, the static pressure coefficient is defined as follows:

\[ C_p = \frac{p_{T1} - p_s}{p_{T1} - \bar{p}_b} \]  

(27)

On the trailing edge circle (red region), the static pressure is roughly constant \((\cong \bar{p}_b)\) except near the trailing edge point on the pressure surface, where a peak in dynamic pressure appears. This is because the velocity is locally accelerated due to an abrupt change in the streamline curvature on the trailing edge point. This indicates that the flow leaving the trailing edge has a deviation angle to the metal trailing edge angle. These features are the same for both Blade A and Blade B.

**Calculated Velocity Distribution**

The free stream velocity at the boundary layer edge \(V_s\) is obtained from the calculated surface static pressure using Eq. (9). Figure 7 compares the normalized surface velocity profiles between Blade A and Blade B at Re = 57,000. The surface velocity is normalized by the velocity \(\bar{V}_b\) defined as follows:

\[ \bar{V}_b = \sqrt{\frac{2(p_{T1} - \bar{p}_b)}{\rho_0}} \]  

(28)

In case of Blade A, the existence of a large laminar separation bubble can be recognized in the surface velocity distribution. Judging from the level of shape factor for calculated suction surface boundary layer at the trailing edge, reattachment of the laminar separation doesn’t seem to have completed before the trailing edge at Re = 57,000.

**Validation of the Assumptions Based on Circulation Theory**

In the present analysis, we evaluate the circulation for a real viscid flow. In the line integral of surface velocity along the blade surface, we use the free stream velocity at the boundary layer edge by Eq. (9) as a surface velocity. However, we use actual blade geometry as an integration path for simplicity. We also use actual blade geometry to define the throat location on the suction surface which form the minimum width of the blade passage. The upstream flow parameters \(V_1\) (or \(V_s\)) and \(\beta_1\) for verification are determined from the flow at the axial location near the inflow boundary, where all flow parameters become uniform.

Firstly, we deal with Eq. (14) based on the assumption that the lift determined by the circulation for a viscid flow is equal to the lift determined by the tangential pressure and viscid drag forces acting on the fluid. The non-dimensional form of Eq. (14) by using an axial velocity is as follows:

\[ \frac{\Gamma_{airfoil}}{V_s S} = \frac{F_{p, y} + F_{d, y}}{\dot{m}V_x} \]  

(29)

Figure 8 shows the percent difference between the left-hand side and the right-hand side of Eq. (29) calculated from the results of RANS simulations for the two blades. The differences are within the range of \(\pm 0.1\%\) over the whole Reynolds numbers.
Secondly, we deal with Eq. (15) based on the assumption that upstream tangential velocity multiplied by the pitch is obtained by subtracting the line integral of suction surface velocity along the path between throat and the trailing edge from the circulation around the airfoil. The non-dimensional form of Eq. (15) by using an axial velocity results in $\tan(\beta_1)$ as follows:

$$
(tan \beta_1)_{circu} = \frac{1}{V_e} \left( \Gamma_{airfoil} - \int_{throat}^{TE} V_{s, suction} ds \right)
$$

(30)

The percent difference between $(tan \beta_1)_{circu}$ from Eq. (30), which is estimated by the concept of circulation, and $(tan \beta_1)_{cfd}$ achieved by the RANS simulation is shown in Fig. 9 for the two blades. Blade A and B show a similar trend of the difference with Reynolds number. The deviation from the true value increases as Reynolds number decreases and reach about 2 percent at Re = 57K for both blades.

![Graph showing percent difference in $\tan(\beta_e)$ vs. Reynolds Number](image)

**Fig. 10 Percent Difference in $\tan(\beta_e)$**

Thirdly, we deal with the assumption that the line integral of velocity tangent to the trailing edge plane $DG$ in Fig. 3 is equal to the line integral of suction surface velocity from throat to the trailing edge as expressed by Eq. (16). From Eq. (10) and (15), the line integral of velocity tangent to the trailing edge plane $DG$ can be rewritten as follows:

$$
\int_{0}^{s_e} v_{te} \sin(\beta_{te} + \alpha_{te}) dt = -V_1 \sin(\beta_1) S + \frac{F_{p,y} + F_{d,y}}{\rho_0 V_e} S
$$

$$
= V_e \sin(\beta_e) S
$$

(31)

As seen in Eq. (31), the line integral of velocity tangent to the trailing edge plane $DG$ is equal to a downstream tangential velocity multiplied by the pitch. Therefore, the non-dimensional forms of Eq. (16) and (31) by using an axial velocity result in $\tan(\beta_e)$. In the present verification, we define $(tan \beta_e)_{circu}$ and $(tan \beta_e)_{cfd}$ as follows:

$$
(tan \beta_e)_{circu} = \frac{1}{V_e} \int_{throat}^{TE} V_{s, suction} ds
$$

(32)

$$
(tan \beta_e)_{cfd} = -\frac{V_1}{V_e} \sin(\beta_1) + \frac{F_{p,y} + F_{d,y}}{m V_e}
$$

(33)

Here, $(tan \beta_e)_{circu}$ is based on the assumption expressed by Eq. (16). And $(tan \beta_e)_{cfd}$ is a verification value of the RANS simulation results, which is obtained by using the calculated upstream velocity near the inflow boundary and the calculated tangential forces of surface pressure and surface skin friction. Figure 10 shows the percent difference between $(tan \beta_e)_{circu}$ and $(tan \beta_e)_{cfd}$ for the two blades. The deviation from the true value increases as Reynolds number decreases and reach about 1 percent at Re = 57K for both blades.

The percent difference in Fig. 8 is very small in a range between -0.1% to +0.1%. On the other hand, the percent differences in Fig. 9 and 10 depend on a Reynolds number, and the deviations reach 1% to 2% at low Reynolds number. This is because $(tan \beta_1)_{circu}$ in Fig. 9 and $(tan \beta_e)_{circu}$ in Fig. 10 include the line integral of suction surface velocity from throat to the trailing edge which is influenced by the definition of throat location, but $(\Gamma_{airfoil})$ in Fig. 8 doesn’t depend on the definition of throat location. In the present analysis, we define the throat location by simply using the actual blade geometry, but it doesn’t correspond to the real throat location that forms the minimum width of the aerodynamical passage. The real throat location varies with boundary layer development, that is, depends on a Reynolds number. The error in the definition of throat location may impact on the accuracy of profile loss estimation. In the following, the resultant accuracy of profile loss estimation will be discussed.

**Comparison of Estimated Profile Losses**

As defined by Denton [5], profile loss is usually taken to be the loss generated in the blade boundary layer and the extra loss arising at a trailing edge. Here, we define the loss that is determined by the pressure and the skin friction drag forces acting on the fluid as the verification profile loss of RANS simulation. Mass conservation equation is:

$$
\dot{m} = \rho_0 V_e \cos(\beta_e) S = \rho_0 V_e \cos(\beta_e) S
$$

(34)

Momentum conservation equation in the axial direction is:

$$
\dot{m} V_e \cos(\beta_e) - \dot{m} V_1 \cos(-\beta_1)
$$

$$
= (p_1 - p_0) S + F_{p,x} + F_{d,x}
$$

(35)

Here, $F_{p,x}$ is the total axial component of the pressure force acting on the fluid per unit span, and $F_{d,x}$ is the total axial component of the skin friction drag force acting on the fluid per unit span. These forces are obtained by using the calculated
surface static pressures and surface shear stresses. Momentum conservation equation in the tangential direction has been already shown in Eq. (11). From Eq. (34) and (35), the downstream mixed-out static pressure is obtained as follows:

$$p_e = \frac{1}{S} \left( F_{p,x} + F_{d,x} \right) + p_1$$  \hspace{1cm} (36)

Then, using the downstream mixed-out velocity obtained from Eq. (11) and (34), the total pressure loss coefficient is expressed as follows:

$$\omega = \frac{\Delta P_T}{\frac{1}{2} \rho_0 V_e^2} = \frac{1}{\frac{1}{2} \rho_0 V_e^2} \left\{ \left( p_T \right)_1 - \left( p_e + \frac{1}{2} \rho_0 V_e^2 \right) \right\}$$  \hspace{1cm} (37)

Here, the verification profile loss expressed by Eq. (37) is referred to as \(\omega_{cfd}\).

The estimation of profile loss for the RANS simulation results is performed through the procedure described before. In the estimation, the numerical solutions at all the grid points on the blade surface are used for the line integrals in Eq. (8), (15) and (22), the calculation of \(F_{p,y}\) and \(F_{d,y}\) in Eq. (21) and the calculation of \(p_b\) in Eq. (22). The boundary layer integral parameters \(\delta_m\) and \(\theta_m\) at the trailing edge points on the suction and pressure surfaces used in Eq. (18) and (20) are also obtained by using the numerical solutions. The estimated profile loss obtained by Eq. (24) is referred to as \(\omega\)_{estimated}.

Figure 11 compares \(\omega\)_{estimated} with \(\omega\)_{cfd} for Blade A and Blade B. Here, it is normalized by \(\omega\)_{cfd} of Blade A at \(Re = 147,000\). The trend of the estimated profile loss with Reynolds number is similar to that of the verification profile loss for both Blade A and Blade B. However, the present method underestimates the level of profile loss for both the two blades. Percent differences between the estimated pressure loss \(\left( \Delta P_T \right)_{estimated}\) and the theoretical pressure loss \(\left( \Delta P_T \right)_{cfd}\) are shown in Fig. 12(a) and 12(b) for Blade A and Blade B, respectively. The underestimations are in a range between -25% to -14%.

In Fig. 12(a) and 12(b), the differences of the estimations by Denton’s method are also compared. The estimations by Denton’s method are performed in two ways. Denton’s method 1 estimates the velocity at the trailing edge plane by using the known mass flow and the calculated boundary layer displacement thicknesses with the metal trailing edge angle. On the other hand, in Denton’s method 2, the velocity and the flow angle at the trailing edge plane, which are estimated by the present method, are used in the profile loss estimation. Therefore, in case of Denton’s method 2, the base pressure loss represented by the first term on the right-hand side of Eq. B-14 in ANNEX B is the same as that in the present estimation. The comparison suggests that the impact of the difference in the flow condition at the trailing edge plane is relatively small for Denton’s method. It seems that the assumption of nearly zero deviation angle used in Denton’s method tends to overestimate a boundary layer loss and a blockage loss. This might result in a better agreement with the verification profile loss for the Denton’s method as seen in Fig. 12(a) and 12(b). However, the resultant profile losses are still underestimated.
**Improved Method for Estimating Profile Loss**

In the boundary layer, the local shear stress at the surface is influenced by a free stream pressure gradient. In case of a turbine, the contribution of a pressure gradient accounts for some part of the skin friction drag force, because the free stream flow on the blade surface undergoes a negative pressure gradient (positive shear stress) over a significant fraction of the surface. This is the reason why the method to use only the boundary layer integral parameters tends to underestimate the profile loss. Therefore, by using Eq. (16) which is relevant to the momentum flux at the trailing edge plane including a free stream flow, we replace the second term in left-hand side of Eq. (5) by the term as follows:

\[
- \frac{\cos(\beta_{te})}{\sin(\beta_{te} + \alpha_e)} \frac{m}{S_e e_{suction}} \int_{Throat}^{T_E} V_s s_{suction} dS
\]

(38)

**Fig. 13 Profile Loss Coefficient by Improved Method**

In Fig. 13, the profile loss coefficients estimated by the improved method \((\omega)_{estimated}\) are compared with the verification profile loss coefficients \((\omega)_{cfd}\) for Blade A and Blade B. Percent differences between the pressure loss estimated by the improved method \((\Delta P_T)_{estimated}\) and the verification pressure loss \((\Delta P_T)_{cfd}\) are shown in Fig. 14(a) and 14(b) for Blade A and Blade B, respectively. The estimations are reasonably improved so that the differences are reduced in a range between -9% to +7%.

**Impact of Modelling for Test Data Analysis**

In the actual low speed cascade test, we study the profile loss with a restricted number of measuring physical quantities and the measurement points. As shown in Fig. 15, the surface static pressures are typically measured at axially 30 locations from 0.025 \(x/C_x\) to 0.975 \(x/C_x\) on the suction surface, 20 locations from 0.025 \(x/C_x\) to 0.975 \(x/C_x\) on the pressure surface and one location on the trailing edge circle in the test at Iwate University. In the region around the leading edge up to 0.025 \(x/C_x\) and the region from 0.975 \(x/C_x\) to the trailing edge circle, there is no information regarding the static pressure distribution. Also, we typically don’t measure surface shear stress to investigate profile loss in a low speed cascade test.

**Fig. 14 Percent Difference in Loss by Improved Method**

**Fig. 15 Surface Pressure Measurement Points**
For the application of the improved estimation method of profile loss to actual low speed cascade test data, it is necessary to model the surface static pressure distribution in the region around the leading edge and around the trailing edge for calculating the pressure forces and the circulation. Figure 16(a) and 16(b) show the methods for modelling the static pressure distribution around the leading edge and around the trailing edge, respectively. As for the absence of surface shear stress data, we neglect the skin friction drag force in Eq. (21) in the profile loss estimation. As a matter of fact, the tangential skin friction drag force to the total tangential force is so small in a range between 0.33% and 0.55% for Blade A and between 0.23% and 0.25% for Blade B over the whole Reynolds numbers according to the RANS simulations.

Fig. 16 Modelling of Surface Pressure Distribution

The impact of the modelling of static pressure distribution and the neglect of viscous force on the accuracy of the profile loss estimation is evaluated by giving the surface static pressures obtained by RANS simulation only at the measurement points in Fig. (15) in the estimation. In Fig. 17, the total pressure loss coefficients estimated by the improved method applied to test data analysis ($\Delta P_T$)estimated and the verification pressure loss ($\Delta P_T$)cfd are shown in Fig. 18(a) and 18(b) for Blade A and Blade B, respectively. The impact is so small in a range up to about one point. It can be expected that the improved method applied to low speed cascade test data estimates the profile loss in a range between -10% to +6% of the real profile loss.
**Estimation of Mass Flow Rate**

The present control volume analysis is capable of estimating not only a mixed-out profile loss but also a mass flow rate. Here, we evaluate the estimation capability by comparing the estimated mass flow rate with the mass flow rate achieved by RANS simulation. The percent differences between the axial velocity estimated by the improved method applied to test data analysis \((V_x)_{estimated}\) and the axial velocity achieved by RANS simulation \((V_x)_{CFD}\) are shown in Fig. 19(a) and 19(b) for Blade A and Blade B, respectively. It can be seen that the improved method applied to test data analysis tends to overestimate the mass flow rate. However, the deviation is about 0.7% at most.

![Fig. 19 Percent Difference in Axial Velocity](image)

**CASCADE TEST DATA ANALYSIS**

The method developed in the present study is applied to the data analysis of low speed cascade tests for the two kinds of blade profiles used in the verification.

---

**Experimental Setup**

A schematic of the test section is shown in Fig. 20. This consists of the linear cascade, a turbulence grid and a moving bar mechanism. There are seven airfoils in the linear cascade, in which two instrumented brass airfoils are included in the middle of the cascade to measure static pressure distribution around the blade surfaces (see Fig. 15). Each of the airfoils, which are designated Blade#3 and Blade#4 in Fig. 20, has 30 pressure holes of 0.5mm diameter on its suction or pressure or trailing edge circle surface.

The moving bar mechanism allows bars to be traversed upstream of the leading edge of the cascade. Cylindrical bars of 3mm diameter (3% axial chord of the cascade) fitted between two timing belts are driven by an inverter-controlled induction motor so that moving wakes are generated at a plane 1.15 axial chords upstream of the cascade.

As shown in Fig. 20, the turbulence grid is installed upstream of the cascade to generate free stream turbulence intensity of 3%. The tests without the turbulence grid are also performed to simulate a low free stream turbulence condition. In case of no turbulence grid, the free stream turbulence intensity is about 0.8%.

![Fig. 20 Test Section of Low Speed Cascade Facility](image)

**Instruments**

Aerodynamic loss profiles at midspan are obtained by measuring inlet and outlet total pressures by use of two miniature Pitot tubes. The outlet measurement is located at 33% \(C_x\) downstream of the trailing edge of the target airfoil. The radius of the probe sensing head is 1.5mm and the radius of the stagnation hole is 0.75mm.

A single hot-wire probe is used for boundary layer measurements. The axial location of the measurements extends from 50% of axial chord to the trailing edge and, in the normal direction to the blade surface, the measurements covered from 0.2% of axial chord (0.2mm) to 10% of axial chord (10mm). A PC-controlled traversing unit, equipped downstream of the
cascade with minimal blockage, enables us to automatically position the probe along a normal line to the airfoil surface.

Uncertainty analysis shows that the inlet velocity error is ±1.7%, the static pressure coefficient is ±3.5% and the hot-wire probe measurements is ±2%.

The detailed description regarding instruments is given by Funazaki et al. [10, 11].

Test Conditions
The steady flow tests are conducted at three different Reynolds numbers, 57,000, 100,000 and 147,000, while the unsteady flow tests are conducted at two different Reynolds numbers, 57,000 and 100,000, with a free stream turbulence intensity of 3% for both Blade A and Blade B cascades. In the unsteady flow, the tests are conducted at three different Strouhal numbers for both Blade A and Blade B cascades. The highest Strouhal number depends on Reynolds number and the type of blade, because the mechanically allowable moving bar speeds are different. In case of no turbulence grid (the free stream turbulence intensity of about 0.8%), the tests are conducted only at Reynolds number of 147,000 with Strouhal numbers, 0 and 0.4, for both Blade A and Blade B cascades.

Results and Discussion

Fig. 21 Comparison of Loss Coefficient In the Steady Flow

Figure 21 compares the estimated loss coefficients relative to the loss coefficient of the verification profile loss obtained by RANS simulation $(\omega)_{cfd}$ of Blade A at $Re = 147,000$. The loss coefficients of estimated profile loss using cascade test data $(\omega)_{test\ data\ estimated}$ are obtained by Eq. (37). In the measured loss coefficients $(\omega)_{measured}$, the total pressure losses measured downstream of the cascades are normalized by the downstream mixed out dynamic pressures obtained by the corresponding profile loss estimations. Both Blade A and Blade B show a similar trend with Reynolds number between the measured pressure loss and the estimated profile loss. However, there are some difference in the loss level over the range of Reynolds number for both Blade A and Blade B.

In Fig. 22(a) and (b), the verification profile losses obtained by RANS simulations $(\omega)_{cfd}$ are plotted for Blade A and Blade B, respectively. It can be observed that the estimated profile losses using cascade test data $(\omega)_{test\ data\ estimated}$ reasonably agree with the verification profile losses except for Blade A at $Re = 57,000$, where the suction surface boundary layer measurement indicates an open separation bubble. It is considered that, in a real flow, the pressure loss is enhanced by a highly unsteady flow of the open separation bubble. It appears that the present estimation captures this enhanced loss. The comparisons in Fig. 22(a) and 22(b) suggest that the present method reasonably estimates the real profile loss. This implies that the pressure loss measured downstream of the cascade includes an additional loss to the profile loss.

Figure 23 shows the percent difference in the ratio of loss coefficient relative to the loss coefficient of the verification profile loss obtained by RANS simulation $(\omega)_{cfd}$ of Blade A at $Re = 147,000$ between the measured pressure loss and the estimated profile loss in the steady flow with a free stream turbulence intensity of 3%. The cases at a lower free stream
turbulence intensity of 0.8% are also plotted in the figure. The level of additional loss in the steady flow seems to depend not only on Reynolds number but also on free stream turbulence intensity.

In Fig. 24(a) and 24(b), the ratio of loss coefficient to the loss coefficient of the verification profile loss obtained by RANS simulation \((\omega)_{cfd}\) of Blade A at \(Re = 147,000\) are compared between the measured pressure loss and the estimated profile loss at \(Re = 57,000\) and \(Re = 100,000\), respectively. It can be seen that, at high Strouhal number, the estimated profile losses for Blade A decrease as the Strouhal numbers increase both at \(Re = 57,000\) and at \(Re = 100,000\). According to the experimental correlation of Coull et al. [4], as the wake passing frequency increases, the separation bubble losses are reduced, because each wake partially suppresses the bubble. It is considered that, for a high suction surface diffusion design of Blade A, a decrease in the separation bubble loss exceeds an increase in the unsteady wake interaction loss at high Strouhal number. On the other hand, the estimated profile losses for Blade B gradually increase as the Strouhal numbers increase both at \(Re = 57,000\) and at \(Re = 100,000\).

Figure 25 shows the percent difference in the ratio of loss coefficient to the loss coefficient of the verification profile loss obtained by RANS simulation \((\omega)_{cfd}\) of Blade A at \(Re = 147,000\) between the measured pressure loss and the estimated profile loss in the unsteady flow with a free stream turbulence intensity of 3%. The cases at a lower free stream turbulence intensity of 0.8% are also plotted in the figure. The pressure loss differences increase with increasing Strouhal number for both Blade A and Blade B. However, Blade A shows a higher rate of increase in the difference. In case of the free stream turbulence intensity of 0.8%, Blade A and Blade B show a similar rate of increase in the difference.

**CONCLUSION**

A method for estimating profile loss of LP turbine blades from the low speed cascade test data is developed. In this method, the flow conditions at the trailing edge plane are derived from the measured velocity distribution on the suction surface from throat to the trailing edge using the concept of circulation.
The validity of the method is verified by using the results of RANS simulations. This method is applied to low speed cascade test data.

1. The use of measured boundary layer integral parameters to represent the momentum deficit at the trailing edge plane in a control volume analysis for estimating the mixed-out downstream flow tends to underestimate the profile loss. Because the boundary layer loss determined by boundary layer integral parameters doesn’t account for that due to a free stream pressure gradient.

2. The representation of the momentum deficit at the trailing edge plane by the flow derived using the concept of circulation, which includes the contribution of a free stream pressure gradient, improves the profile loss estimation. It is expected that, in case of a low speed cascade test data analysis, the accuracy is in a range between -10% to +6%.

3. The comparison between the present estimation for cascade test data and the measurement implies that the measured total pressure loss downstream of the cascade includes an additional loss to the profile loss even in the steady flow. This indicates that there are other sources of a pressure loss generated outside the boundary layer and/or downstream of the trailing edge except a mixing loss of the profile loss even in the steady flow.

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REFERENCES


ANNEX A

MODELLING OF FLOW AT TRAILING EDGE PLANE

Fig. A-1 Flow Model at Trailing Edge Plane

It is assumed that the flow angle along the trailing edge plane $AB$ is uniform with the angle $\beta_{te}$ and the boundary layer flows normal to the average flow angle $\beta_{te}$ on the trailing edge are kept the same as the flows along the trailing edge plane. The amount of mass flow across the plane $AB$ is:

$$
\int_0^{S_e} \rho_0 v_{te} \cos(\beta_{te} + \alpha_e) \, dt = \rho_0 \int_0^{W_e} v_{te} \, dw \quad (A-1)
$$

Here,

$$
S_e = \sqrt{(x_A - x_B)^2 + (y_A + y_B)^2} \quad (A-2)
$$

$$
W_e = S_e \cos(\beta_{te} + \alpha_e) \quad (A-3)
$$

$$
dt = \frac{dw}{\cos(\beta_{te} + \alpha_e)} \quad (A-4)
$$

Using the following definition of displacement thickness,

$$
\rho_0 V_{te} \delta = \int_0^{W_e} \rho_0 (V_{te} - v_{te}) \, dw
$$

$$
= \rho_0 V_{te} W_e - \rho_0 \int_0^{W_e} v_{te} \, dw \quad (A-5)
$$

The mass flow is expressed as follows:

$$
\dot{m} = \rho_0 V_{te} S_e \cos(\beta_{te} + \alpha_e) \left(1 - \frac{\delta}{W_e}\right) \quad (A-6)
$$

The flow rates of momentum across the plane $AB$ in the axial and tangential directions are:

$$
\int_0^{S_e} \rho_0 v_{te} \cos(\beta_{te} + \alpha_e) \, v_{te} \cos(\beta_{te}) \, dt
$$

$$
= \rho_0 \cos(\beta_{te}) \int_0^{W_e} v_{te}^2 \, dw \quad (A-7)
$$

$$
\int_0^{S_e} \rho_0 v_{te} \cos(\beta_{te} + \alpha_e) \, v_{te} \sin(\beta_{te}) \, dt
$$

$$
= \rho_0 \sin(\beta_{te}) \int_0^{W_e} v_{te}^2 \, dw \quad (A-8)
$$

Using the following definition of momentum thickness,

$$
\rho_0 V_{te}^2 \theta = \int_0^{W_e} \rho_0 v_{te} (V_{te} - v_{te}) \, dw
$$

$$
= \rho_0 V_{te}^2 W_e \left(1 - \frac{\delta}{W_e}\right) - \rho_0 \int_0^{W_e} v_{te}^2 \, dw \quad (A-9)
$$

Equation (A-7) and (A-8) can be expressed as follows:

$$
\int_0^{S_e} \rho_0 v_{te} \cos(\beta_{te} + \alpha_e) \, v_{te} \cos(\beta_{te}) \, dt
$$

$$
= \dot{m} V_{te} \cos(\beta_{te}) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}} \quad (A-10)
$$

$$
\int_0^{S_e} \rho_0 v_{te} \cos(\beta_{te} + \alpha_e) \, v_{te} \sin(\beta_{te}) \, dt
$$

$$
= \dot{m} V_{te} \sin(\beta_{te}) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}} \quad (A-11)
$$

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ANNEX B

DERIVATION OF DENTON’S MIXING EQUATION

Fig. B-1 Control Volume for Mixing Analysis

It is assumed that the flow on the trailing edge plane AB is kept the same until the trailing edge tangential plane, and also assumed that the difference between \( \beta_{te} \) and \( \beta_e \) is small.

Mass conservation

\[
\rho_0V_{te} \cos(\beta_{te}) S \left(1 - \frac{t + \delta}{W_e}\right) = \rho_0V_e \cos(\beta_e) S
\]  \hspace{1cm} (B-1)

\[W_e = S \cos(\beta_{te})\]  \hspace{1cm} (B-2)

Momentum conservation in the axial direction

\[
\rho_0V_{e}^2(\cos\beta_e)^2 S - \rho_0V_{te}^2(\cos\beta_{te})^2 S \left(1 - \frac{t + \delta}{W_e} - \frac{\theta}{W_e}\right)
= (p_{te} - p_e) S + (p_b - p_{te}) \frac{t}{\cos\beta_{te}}
\]  \hspace{1cm} (B-3)

Momentum conservation in the tangential direction

\[
\rho_0V_e^2 \cos(\beta_e) \sin(\beta_e) S
- \rho_0V_{te}^2 \cos(\beta_{te}) \sin(\beta_{te}) S \left(1 - \frac{t + \delta}{W_e} + \frac{\theta}{W_e}\right) = 0
\]  \hspace{1cm} (B-4)

From Eq. (B-1), the following equation is obtained:

\[
\left(\frac{V_e}{V_{te}}\right)^2 = \left(\frac{\cos(\beta_{te})}{\cos(\beta_e)}\right)^2 \left(1 - \frac{t + \delta}{W_e}\right)^2
\]  \hspace{1cm} (B-5)

While, from Eq. (B-4), another equation is obtained:

\[
\left(\frac{V_e}{V_{te}}\right)^2 = \frac{\cos(\beta_{te}) \sin(\beta_{te})}{\cos(\beta_e) \sin(\beta_e)} \left(1 - \frac{t + \delta}{W_e} - \frac{\theta}{W_e}\right)
\]  \hspace{1cm} (B-6)

From Eq. (B-5) and (B-6),

\[\tan(\beta_{te}) \left(1 - \frac{t + \delta}{W_e} - \frac{\theta}{W_e}\right) = \tan(\beta_e) \left(1 - \frac{t + \delta}{W_e}\right)^2\]  \hspace{1cm} (B-7)

Here, \( \beta_e = \bar{\beta}_{te} + \epsilon \), and \( \epsilon \) is assumed to be small. Then,

\[\tan(\beta_e) = \tan(\beta_{te}) \left(1 + \epsilon \frac{1}{\sin(\beta_{te}) \cos(\beta_{te})}\right)\]  \hspace{1cm} (B-8)

Using Eq. (B-8), Eq. (7) can be rewritten as follows:

\[\epsilon \left(1 - \frac{t + \delta}{W_e}\right)^2 = \sin(\beta_{te}) \cos(\beta_{te}) \left(t + \delta - \frac{(t + \delta)^2}{W_e} - \frac{\theta}{W_e}\right)\]  \hspace{1cm} (B-9)

From the assumption that \( \epsilon \) is small,

\[\frac{1}{\cos^2(\beta_e)} = \frac{1}{\cos^2(\beta_{te})} \left(1 + 2\epsilon \tan(\beta_{te})\right)\]  \hspace{1cm} (B-10)

Using Eq. (B-9) and (B-10), Eq. (B-5) results in the following:

\[\left(\frac{V_e}{V_{te}}\right)^2 = \left(1 - \frac{t + \delta}{W_e}\right)^2 + 2(\sin(\beta_{te}))^2 \left(\frac{t + \delta}{W_e} - \frac{(t + \delta)^2}{W_e} - \frac{\theta}{W_e}\right)\]  \hspace{1cm} (B-11)

The total pressure loss coefficient is:

\[
\frac{\Delta P_T}{\frac{1}{2} \rho_0 V_{te}^2} = -\frac{p_b - p_{te}}{\frac{1}{2} \rho_0 V_{te}^2} + 1 - \left(\frac{V_e}{V_{te}}\right)^2
\]  \hspace{1cm} (B-12)

From Eq. (B-3), the following equation is obtained:

\[
\frac{p_{te} - p_e}{\frac{1}{2} \rho_0 V_{te}^2} = -\frac{p_b - p_{te}}{\frac{1}{2} \rho_0 V_{te}^2} \frac{t}{W_e}
+ 2(\cos(\beta_{te})^2 \left(\frac{t + \delta}{W_e} + \frac{(t + \delta)^2}{W_e} + \frac{\theta}{W_e}\right)\right)\]  \hspace{1cm} (B-13)

Using Eq. (B-11) and (B-13), Eq. (B-12) is as follows:

\[
\frac{\Delta P_T}{\frac{1}{2} \rho_0 V_{te}^2} = -c_{pb} \left(\frac{t}{W_e}\right) + \left(\frac{t + \delta}{W_e}\right)^2 + 2\frac{\theta}{W_e}
\]  \hspace{1cm} (B-14)
Fig. C-1 Control Volume for Mixing Analysis

Consider a mixed-out flow on the trailing edge plane $AB$. As shown in Fig. C-1, the nonuniform flow is mixed-out on the plane $AB$ as a mixing plane so that the mass and the momentum are conserved across the mixing plane.

Mass conservation

$$m = \rho_0 V_{te} W_e \left( 1 - \frac{\delta}{W_e} \right) = \rho_0 \bar{V}_m \cos(\bar{\beta}_m + \alpha_e) S_e \quad (C-1)$$

Momentum conservation in the axial direction

$$m \bar{V}_m \cos(\bar{\beta}_m) - m V_{te} \cos(\bar{\beta}_{te}) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}}$$

$$= (\bar{p}_{te} - \bar{p}_m) S_e \cos \alpha_e \quad (C-2)$$

Momentum conservation in the tangential direction

$$m \bar{V}_m \sin(\bar{\beta}_m) - m V_{te} \sin(\bar{\beta}_{te}) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}}$$

$$= -(\bar{p}_{te} - \bar{p}_m) S_e \sin \alpha_e \quad (C-3)$$

From Eq. (C-1), (C-2) and (C-3), the following equation can be obtained:

$$\bar{V}_m \sin(\bar{\beta}_m + \alpha_e) = V_{te} \sin(\bar{\beta}_{te} + \alpha_e) \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}} \quad (C-4)$$

On the other hand, by applying Eq. (16) to the mixed-out flow, another equation can be obtained:

$$\bar{V}_m \sin(\bar{\beta}_m + \alpha_e) S_e = \int_{Throat}^{TE} V_{s,suction} ds \quad (C-5)$$

From Eq. (C-4) and (C-5), The following relation is obtained:

$$V_{te} \sin(\bar{\beta}_{te} + \alpha_e) S_e \frac{1 - \frac{\delta}{W_e} - \frac{\theta}{W_e}}{1 - \frac{\delta}{W_e}}$$

$$= \int_{Throat}^{TE} V_{s,suction} ds \quad (C-6)$$